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The supersymmetric modified Pöschl–Teller and delta well potentials

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Abstract. New supersymmetric (SUSY) partners of the modified Pöschl–Teller and the Dirac’s delta well potentials are constructed in closed form. The resulting one-parametric potentials are shown to be interrelated by a limiting process. The range of values of the parameters for which these potentials are free of singularities is exactly determined. The construction of higher-order SUSY partner potentials is also investigated.

1. Introduction

The modified Pöschl–Teller potential $V(\alpha, x) = -U_0(\cosh \alpha x)^{-2}$, is one of the few exactly solvable potentials in quantum mechanics. It was first analysed by Rosen and Morse [1], who found its energy eigenvalues and eigenfunctions. (This potential $V(\alpha, x)$ is, in fact, a hyperbolic version of what is commonly known as the Pöschl–Teller potential [2].) The system is characterized by a finite number of bound states whose spectrum depends on the parameters $U_0, \alpha > 0$, plus a continuum of scattering states (see [3] for an elegant group approach); the normalization of the wavefunctions has been obtained by Nieto [4]. Barut *et al* [5] studied a three-dimensional version of the problem (with an additional term proportional to $(\sinh \alpha x)^{-2}$ in the potential, which was already present in [2]), and using the Infeld–Hull factorization [6] they constructed ladder operators to determine the bound and scattering states from the matrix elements of group representations. It is well known that the modified Pöschl–Teller potential appears in the solitary wave solutions of the Korteweg–de Vries equation [7], and also that it can be obtained from supersymmetric (SUSY) quantum mechanics [8] as the SUSY partner of the free particle potential [9], being a nontrivial example of unbroken SUSY [10].

This potential can be considered as belonging to the type of short-range potentials, which almost vanish over most of their domain, except near zero (where the source resides). The extreme case of such short-range potentials is the Dirac delta well $V(x) = -g\delta(x)$, $g > 0$. Delta potentials have long been used in field theories, where the main problem arises from the regularization and renormalization of the values for the physical observables predicted by the theory itself [11]. The effects of adding a delta function potential to the states of a previously known potential have been computed exactly by Atkinson and Crater [12] for several physical systems. The one-dimensional problem of N particles interacting by means of delta potentials is one of the simplest many-body problems solved exactly (see [13] and references therein). As the delta potential $\delta(x)$ is a distribution rather than a function, it can be approximated by

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different families of functions; one of them ($V(\alpha, x)$ with an appropriate choice of the depth U_0) will be investigated in this paper.

Concerning the study of new exactly solvable problems in quantum mechanics, in recent years there has been remarkable progress along different lines: Darboux transformation [14], Infeld–Hull factorization [6], Mielnik factorization [15–17], SUSY quantum mechanics [8] and inverse scattering theory [18], among others. It is worth stressing that all of them can be embraced in an elegant algebraic approach named *intertwining technique* [19], which has been successfully applied in the construction of higher-order SUSY partners [20, 21]. The usefulness of the intertwining has also been proved in the study and interpretation of black-hole perturbations in general relativity [22].

The main purpose of this paper is to analyse the SUSY partners associated with the modified Pöschl–Teller and Dirac delta potentials by using the intertwining technique. In section 2, after a short review of results concerning both potentials, the properties of the latter are straightforwardly obtained as a limiting case of the former. In section 3 we will determine closed expressions for two new families of SUSY partner potentials of Pöschl–Teller, paying attention to the appearance of singularities (in that case the isospectral properties should be considered under the point of view of [23]). Besides, the limit already mentioned in section 2 is carefully studied here for these two families, obtaining very different behaviour for each of them. A direct analysis of the SUSY delta potential is also carried out to check the validity of the previous limits. Finally, in section 4 other varieties of Pöschl–Teller SUSY potentials are shown to be easily obtained using higher-order intertwining.

2. The essentials of the modified Pöschl–Teller potential

Let us consider the well known one-dimensional two-parametric modified Pöschl–Teller potential [24], written in the following equivalent forms:

$$V(\alpha, x) = -\frac{U_0}{\cosh^2 \alpha x} = -\frac{\hbar^2}{2m} \alpha^2 \frac{\lambda(\lambda - 1)}{\cosh^2 \alpha x} = -\frac{g\alpha}{2 \cosh^2 \alpha x} \quad \alpha > 0. \quad (1)$$

Henceforth, in all three versions, the following conditions are imposed in order to have an attractive potential: $U_0 > 0$, $\lambda > 1$ or $g > 0$. We also take for simplicity $\hbar^2/2m = 1$, and from the last equality the parameters α , λ and g are related by

$$\lambda = \frac{1}{2} \left(1 + \sqrt{1 + \frac{2g}{\alpha}} \right) > 1. \quad (2)$$

The bound states ($E < 0$) for this potential can be obtained by using the traditional recipe of transforming the stationary Schrödinger equation into a hypergeometric equation, with parameters

$$a = \frac{1}{2} \left(\lambda - \frac{\sqrt{|E|}}{\alpha} \right) \quad b = \frac{1}{2} \left(\lambda + \frac{\sqrt{|E|}}{\alpha} \right) \quad c = \frac{1}{2}. \quad (3)$$

The general solution is

$$\psi(x) = (\cosh \alpha x)^\lambda [A {}_2F_1(a, b; \frac{1}{2}; -\sinh^2 \alpha x) + B(\sinh \alpha x) {}_2F_1(a + \frac{1}{2}, b + \frac{1}{2}; \frac{3}{2}; -\sinh^2 \alpha x)]. \quad (4)$$

The normalization condition on these eigenfunctions allows us to determine the energy spectrum, which is found to be

$$E_n = -\alpha^2(\lambda - 1 - n)^2 \quad n \in \mathbb{N}, \quad 0 \leq n < \lambda - 1 \quad (5)$$

or, just in terms of α and g :

$$E_n = -\alpha^2 \left(\frac{1}{2} \sqrt{1 + \frac{2g}{\alpha}} - \frac{1}{2} - n \right)^2 \quad n = 0, 1, 2, \dots < \frac{1}{2} \sqrt{1 + \frac{2g}{\alpha}} - \frac{1}{2}. \quad (6)$$

From (5) and $\lambda > 1$, the energy for $n = 0$ always belongs to the spectrum of $V(\alpha, x)$. Calling N the biggest possible value of n in (6), the total number of bound states is $N + 1$. From there, the following relationship between the parameters α and N holds:

$$\frac{2g}{(2N + 3)^2 - 1} \leq \alpha < \frac{2g}{(2N + 1)^2 - 1}.$$

If $N = 0$, then $\alpha \geq g/4$, and there is just one bound state such that it has the lowest energy E_0 , and for which we have

$$a_0 = c_0 = \frac{1}{2} \quad b_0 = \lambda - \frac{1}{2} \quad A \neq 0 \quad B = 0 \quad E_0 = -\alpha^2(\lambda - 1)^2. \quad (7)$$

The corresponding normalized wavefunction turns out to be

$$\psi_0(x) = C_0(\cosh \alpha x)^\lambda {}_2F_1 \left(\frac{1}{2}, \lambda - \frac{1}{2}; \frac{1}{2}; -\sinh^2 \alpha x \right) = \sqrt{\frac{\alpha \Gamma(\lambda - \frac{1}{2})}{\sqrt{\pi} \Gamma(\lambda - 1)}} (\cosh \alpha x)^{1-\lambda}. \quad (8)$$

It is also interesting to remark that (1) is a *transparent* potential (i.e. the reflexion coefficient is equal to zero) when the following condition is verified [24]:

$$g = 2\alpha k(k + 1) \quad k = 0, 1, 2, \dots \quad (9)$$

Using our notation this implies

$$\lambda = k + 1 \quad k = 1, 2, \dots \quad (10)$$

When $k = 0$ we have the case of the free particle, which was excluded from the very beginning.

The last form of the potential $V(\alpha, x)$ given in (1) can be directly related to the Dirac delta potential $V_D(x) = -g\delta(x)$ simply by taking the limit $\alpha \rightarrow \infty$:

$$\lim_{\alpha \rightarrow \infty} V(\alpha, x) = -g\delta(x). \quad (11)$$

As is well known, this delta well has a unique bound state with eigenvalue

$$E_\delta = -(g/2)^2 \quad (12)$$

and normalized wavefunction

$$\psi_\delta(x) = \sqrt{g/2} e^{-g|x|/2}. \quad (13)$$

These results can be obtained directly from the modified Pöschl–Teller potential by proving that the limiting relationship (11) between both types of potential is also inherited by their eigenfunctions and energy eigenvalues. Indeed, from the Pöschl–Teller ground state energy level (7) we have

$$\lim_{\alpha \rightarrow \infty} E_0 = -\lim_{\alpha \rightarrow \infty} \frac{\alpha^2}{4} \left(\frac{g}{\alpha} - \frac{g^2}{2\alpha^2} + \dots \right)^2 = -\left(\frac{g}{2} \right)^2 = E_\delta. \quad (14)$$

A similar analysis can be done for the ground state eigenfunction (8). Observe that $\psi_0(0) = C_0$; if $x \neq 0$, then the limit $\alpha \rightarrow \infty$ is equivalent to $\alpha x \rightarrow \pm\infty$, hence it can be calculated from the asymptotic form of the function. To find it, we use

$$\cosh \alpha x \sim \frac{e^{\alpha|x|}}{2} \quad \sinh \alpha x \sim \pm \frac{e^{\alpha|x|}}{2} \quad (\alpha > 0) \quad (15)$$

and also the fact that, for $z \rightarrow -\infty$, we have [25]

$${}_2F_1(a, b; c; z) \sim \Gamma(c) \left\{ \frac{\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} + \frac{\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} \right\}$$

$$|\arg(-z)| < \pi. \tag{16}$$

Then, from (7) and (16), we have for $|z| \rightarrow \infty$

$${}_2F_1\left(\frac{1}{2}, b_0; \frac{1}{2}; -z^2\right) \sim z^{-2b_0}. \tag{17}$$

In addition, we have the following asymptotic behaviour for $\alpha \rightarrow \infty$: $2b_0 \sim 1 + g/\alpha$. Hence

$$\lim_{\alpha \rightarrow \infty} \psi_0(x) = C_\infty \lim_{\alpha \rightarrow \infty} 2^{b_0 - \frac{1}{2}} e^{-|x|\alpha(b_0 - \frac{1}{2})} = C_\infty e^{-g|x|/2} \tag{18}$$

where $C_0 \rightarrow C_\infty$. From (8) it is clear that $C_\infty = \sqrt{g/2}$, and then equation (18) states the connection between (8) and (13) when the limiting relationship (11) is satisfied.

Finally, it is also well known that the delta well potential is transparent. This property is also deduced from the Pöschl–Teller potential by taking the limit $\alpha \rightarrow \infty$ in (10), and using (2), which gives the solution $k = 0$.

3. The SUSY modified Pöschl–Teller potential

Let us consider now the problem of finding the SUSY partner of the modified Pöschl–Teller potential $V(\alpha, x)$. We look for a first-order differential operator $A = \frac{d}{dx} + \beta(x)$ and a partner potential $\tilde{V}(\alpha, x)$ such that the following intertwining relationship holds:

$$\left[-\frac{d^2}{dx^2} + \tilde{V}(\alpha, x) \right] A = A \left[-\frac{d^2}{dx^2} + V(\alpha, x) \right]. \tag{19}$$

The new potential $\tilde{V}(\alpha, x)$ is related to $V(\alpha, x)$ through the following SUSY relationship:

$$\tilde{V}(\alpha, x) = V(\alpha, x) + 2\beta'(x) \tag{20}$$

where $\beta(x)$ is a solution of the Riccati equation

$$\beta^2(x) - \beta'(x) = V(\alpha, x) - \epsilon \tag{21}$$

with ϵ an integration constant, which turns out to be the factorization energy. There is an immediate particular solution of equation (21) in the form of a hyperbolic tangent, $\beta_0 = D \tanh \alpha x$, with D depending on α . The introduction of β_0 in (21) gives

$$D^+ = -\alpha\lambda \quad D^- = -\alpha(1 - \lambda). \tag{22}$$

Therefore, we have two different particular solutions of (21):

$$\beta_0^\pm(\alpha, x) = D^\pm \tanh \alpha x \tag{23}$$

associated with two different factorization energies

$$\epsilon^\pm = -(D^\pm)^2 = -\frac{\alpha^2}{2} \left(1 + \frac{g}{\alpha} \pm \sqrt{1 + \frac{2g}{\alpha}} \right). \tag{24}$$

Note that these factorization energies can be formally identified with two values of the spectrum formula (6):

$$\epsilon^- = E_0 \quad \epsilon^+ = E_{-1}. \tag{25}$$

Then, the general solutions of the Riccati equation (21) for the above factorization energies can be found to be

$$\beta_\zeta^+(\alpha, x) = D^+ \tanh \alpha x - \frac{d}{dx} \ln \left(1 - \zeta \int^x (\cosh \alpha y)^{2D^+/\alpha} dy \right) \quad (26)$$

$$\beta_\xi^-(\alpha, x) = D^- \tanh \alpha x - \frac{d}{dx} \ln \left(1 - \xi \int^x (\cosh \alpha y)^{2D^-/\alpha} dy \right) \quad (27)$$

where ζ, ξ are two new independent integration constants; when they are taken to be zero, we recover the particular solutions $\beta_0^\pm(x)$. It must be clear that, in fact, we have obtained two different families of intertwining operators

$$A_\zeta^+ = \frac{d}{dx} + \beta_\zeta^+(x) \quad A_\xi^- = \frac{d}{dx} + \beta_\xi^-(x) \quad (28)$$

generating two different families of SUSY partners (20) of the potential (1). Until now, it has been usual to consider only the SUSY partners of a given potential constructed by taking particular solutions of the Riccati equation [8,9]. For the potential we are dealing with, the cases $\zeta = 0$ or $\xi = 0$ give interesting results, and can be obtained simply as byproducts of (26), (27). In principle, it is possible to find solutions associated to other factorization energies, but they produce very awkward expressions, without adding new relevant information to the problem under consideration.

Let us now analyse the results emerging when the general solutions (26), (27) are taken into account. Observe that the integrals appearing there can be expressed in a closed form as

$$\int^x (\cosh \alpha y)^q dy = -\frac{2^{-q} e^{-\alpha q x}}{\alpha q} {}_2F_1 \left(-\frac{q}{2}, -q; 1 - \frac{q}{2}; -e^{2\alpha x} \right) + \text{constant}. \quad (29)$$

This expression will be used next to construct the closed form of the SUSY partner potentials.

3.1. The two-parametric family of potentials $\tilde{V}_\zeta^+(\alpha, x)$

In this case, taking into account (22) and (26), we see that the exponent in (29) is negative, indeed $q = -2\lambda$. The definite integral exists in the whole real axis, and we can define the function

$$M(\lambda, \alpha, x) = \int_0^x (\cosh \alpha y)^{-2\lambda} dy = \frac{2^{2\lambda} e^{2\alpha\lambda x}}{2\alpha\lambda} {}_2F_1(\lambda, 2\lambda; 1 + \lambda; -e^{2\alpha x}) - \frac{\sqrt{\pi} \Gamma(\lambda)}{2\alpha \Gamma(\lambda + \frac{1}{2})}. \quad (30)$$

A typical plot of $M(\lambda, \alpha, x)$ is shown in figure 1 for general values of the parameters α and λ . It is quite clear that this function is odd in the variable x , and is monotonically increasing from its minimum value $M(\lambda, \alpha, -\infty) = -\sqrt{\pi} \Gamma(\lambda)/(2\alpha \Gamma(\lambda + \frac{1}{2}))$ to its maximum value $M(\lambda, \alpha, +\infty) = |M(\lambda, \alpha, -\infty)|$. It is interesting to remark on the resemblance between the form of this function and that of the error function $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$. We will immediately see that, indeed, for the Pöschl–Teller problem, $M(\lambda, \alpha, x)$ plays a role completely analogous to that played by the error function when determining the one-parametric family of SUSY partners of the harmonic oscillator potential [15].

If we introduce now the function

$$\Omega_\zeta^+(\lambda, \alpha, x) := \frac{(\cosh \alpha x)^{-2\lambda}}{1 - \zeta M(\lambda, \alpha, x)} = -\frac{1}{\zeta} \frac{d}{dx} \ln [1 - \zeta M(\lambda, \alpha, x)] \quad (31)$$

we have for β^+ in (26) the following expression:

$$\beta_\zeta^+(\alpha, x) = -\alpha \lambda \tanh \alpha x + \zeta \Omega_\zeta^+(\lambda, \alpha, x) = \frac{d}{dx} \ln [(\cosh \alpha x)^\lambda \Omega_\zeta^+(\lambda, \alpha, x)]. \quad (32)$$

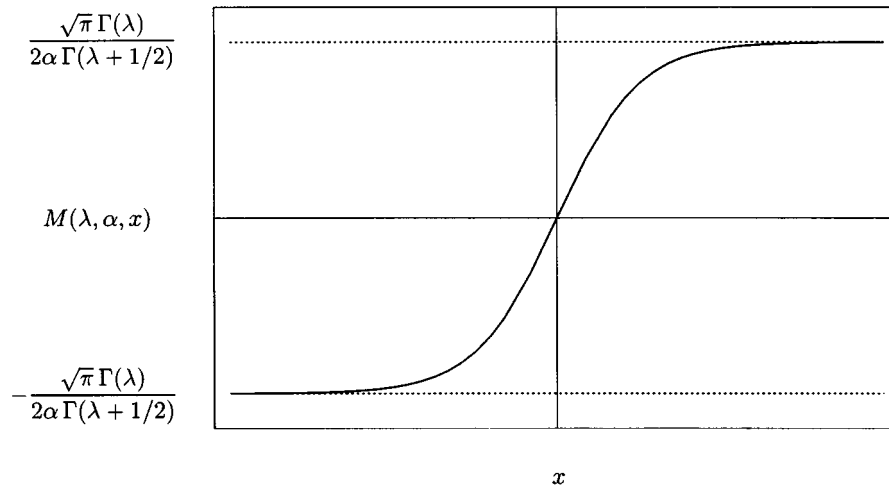


Figure 1. The function $M(\lambda, \alpha, x)$ given in equation (30). Its bounded character enables the existence of nonsingular SUSY partners $\tilde{V}_\zeta^+(\alpha, x)$.

From here, and using (20), we can evaluate the associated SUSY partner potential, which turns out to be

$$\tilde{V}_\zeta^+(\alpha, x) = -\alpha^2 \left(1 + \frac{g}{2\alpha} + \sqrt{1 + \frac{2g}{\alpha}} \right) \frac{1}{\cosh^2 \alpha x} - 4\lambda\alpha\zeta\Omega_\zeta^+(\lambda, \alpha, x) \tanh \alpha x + 2(\zeta\Omega_\zeta^+(\lambda, \alpha, x))^2. \quad (33)$$

It is obvious that the singularities of \tilde{V}_ζ^+ correspond to the singular points of the function $\Omega_\zeta^+(\lambda, \alpha, x)$. It can be proved that this function is free of singularities in the following range of values of ζ :

$$|\zeta| < \frac{1}{M(\lambda, \alpha, +\infty)} = \frac{2\alpha\Gamma(\lambda + \frac{1}{2})}{\sqrt{\pi}\Gamma(\lambda)}. \quad (34)$$

As we pointed out before, if we compare our results with Mielnik's pioneering work on the harmonic oscillator [15], one can appreciate that the roles played there by the error function and his parameter γ , are performed here by $M(\lambda, \alpha, x)$ and the inverse of ζ . The characteristic features of $M(\lambda, \alpha, x)$ determine the existence of SUSY partner potentials which are free of singularities, and are therefore almost isospectral to the modified Pöschl–Teller potential. (A similar analysis can be done for an equivalent integral appearing in [16].)

Let us now work in this range of values of the parameter ζ . The potential (33) corresponds to the following family of almost isospectral Hamiltonians:

$$\tilde{H}_\zeta^+ := -\frac{d^2}{dx^2} + \tilde{V}_\zeta^+(\alpha, x) = A_\zeta^+(A_\zeta^+)^{\dagger} + \epsilon^+. \quad (35)$$

It is well known that the eigenfunctions of \tilde{H}_ζ^+ can be constructed by acting with the operator A_ζ^+ of (28) on the eigenfunctions ψ_n of H , which is factorized as

$$H = -\frac{d^2}{dx^2} + V(\alpha, x) = (A_\zeta^+)^{\dagger} A_\zeta^+ + \epsilon^+ \quad (36)$$

and are given by $\tilde{\psi}_n^+(\zeta, x) \propto A_\zeta^+ \psi_n(x)$, provided that $A_\zeta^+ \psi_n(x) \neq 0$ and $\tilde{\psi}_n^+(\zeta, x) \in L^2(\mathbb{R})$. There is also the possibility of an extra eigenfunction $\tilde{\varphi}^+(\zeta, x)$ satisfying

$$(A_\zeta^+)^{\dagger} \tilde{\varphi}^+(\zeta, x) = 0 \quad (37)$$

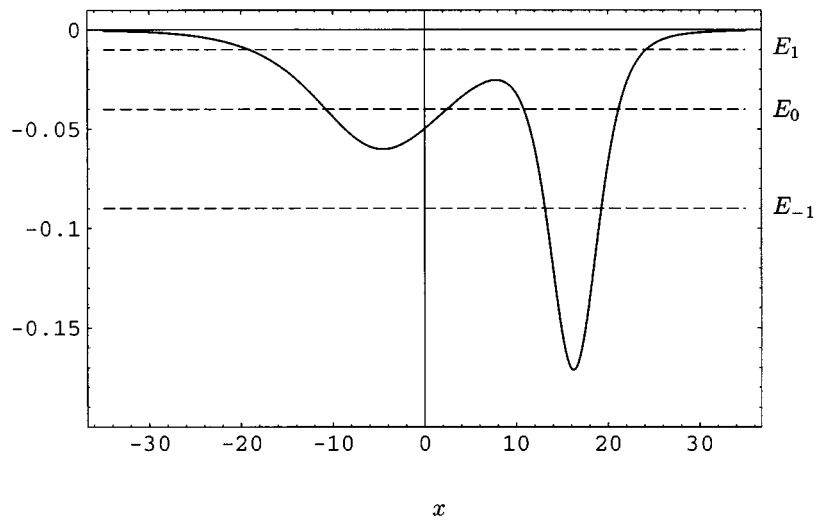


Figure 2. The SUSY partner potential $\tilde{V}_\zeta^+(\alpha, x)$ given by equation (33), with $\alpha = 0.1$, $\lambda = 3$, and $\zeta = 0.0937$, is an asymmetric double well. Its three bound states are represented with dotted horizontal lines.

which we call the ‘missing state’.

The first point we want to stress is that, according to (25), $\epsilon^+ = E_{-1}$ which is an energy level not allowed in the spectrum of the initial Hamiltonian H . Hence, from (36) it is clear that there is no eigenfunction $\hat{\psi}$ of H annihilated by A_ζ^+ . Therefore, the eigenfunctions of \tilde{H}_ζ^+ are given by the normalized functions

$$\tilde{\psi}_n^+(\zeta, x) = (E_n - \epsilon^+)^{-1/2} A_\zeta^+ \psi_n(x) \quad n = 0, 1, \dots \tag{38}$$

plus the missing state solving (37) which, properly normalized, reads

$$\tilde{\varphi}^+(\zeta, x) = \sqrt{\frac{1 - \zeta^2 M^2(\lambda, \alpha, +\infty)}{2M(\lambda, \alpha, +\infty)}} (\cosh \alpha x)^\lambda \Omega_\zeta^+(\lambda, \alpha, x). \tag{39}$$

Note that the non-singularity condition (34) appears here again, although in this case it is required for the missing state to be normalizable. Note, also, from (35) that $\tilde{\varphi}^+(\zeta, x)$ is clearly the eigenfunction of \tilde{H}_ζ^+ with eigenvalue ϵ^+ . This is why we named it the missing state. The spectrum of \tilde{H}_ζ^+ is given by the set $\{E_n; n = 0, 1, \dots\}$ plus a new level at $\epsilon^+ = E_{-1}$. The conclusion is immediate: the family of potentials $\tilde{V}_\zeta^+(\alpha, x)$ is not strictly isospectral to its SUSY partner $V(\alpha, x)$: it has the same levels plus an additional one which is placed below all of them. Let us remark that, due to the annihilation of the missing state $\tilde{\varphi}^+(\zeta, x)$ in (39) by the intertwiner A_ζ^+ , the missing state has no SUSY partner, and the couple of almost isospectral Hamiltonians H and H_ζ^+ corresponds to a case of unbroken SUSY.

In figure 2 we have plotted the asymmetric double well corresponding to the SUSY partner potential $\tilde{V}_\zeta^+(\alpha, x)$ given by (33), with $\alpha = 0.1$, $\lambda = 3$, and $\zeta = 0.0937$. The three bound states of this potential are represented with dotted horizontal lines. Note that the potentials $\tilde{V}_\zeta^+(\alpha, x)$ present features that make them interesting for physical applications: (i) we are able to know their spectra in an exact form, and by adjusting the parameters we can have the desired number of bound states, and (ii) the shape of the potential can be modified to allow interesting tunnelling effects.

Let us now consider the limit $\alpha \rightarrow \infty$ of the potentials (33). There are two terms containing the function $\Omega_{\zeta}^+(\lambda, \alpha, x)$; using (15), (16), and (34), it can be easily proved that this has the following behaviour for large values of α :

$$\Omega_{\zeta}^+(\lambda, \alpha, x) \xrightarrow{\alpha \rightarrow \infty} \begin{cases} 4e^{-2\alpha|x|} & x \neq 0 \\ 1 & x = 0 \end{cases} \quad (40)$$

giving a discontinuous function in the limit, but which is zero almost everywhere. In addition, the product $\alpha\Omega_{\zeta}^+(\lambda, \alpha, x) \rightarrow 4\delta(x)$, and $\alpha\Omega_{\zeta}^+(\lambda, \alpha, x) \tanh \alpha x \rightarrow 0$. Hence, the only relevant part in this potential would be that coming from the first term. But it diverges very badly as $-4\alpha\delta(x)$, and therefore we do not end up with a physically interesting potential.

3.2. The two-parametric family of potentials $\tilde{V}_{\xi}^-(\alpha, x)$

The study of the other family of potentials, derived from (27), can be done according to the lines already followed in section 3.1. Nevertheless, there are important differences between the results obtained in both cases. First of all, from equations (22) and (27), the exponent in (29) is now positive: $q = 2(\lambda - 1)$. As a consequence, if we try to evaluate the integral in the whole real axis we get a divergent result. Nevertheless, it is useful to introduce a function similar to $M(\lambda, \alpha, x)$: let us call it

$$\begin{aligned} L(\lambda, \alpha, x) &= \int_0^x (\cosh \alpha y)^{2(\lambda-1)} dy \\ &= -\frac{e^{-2\alpha(\lambda-1)x}}{2^{2(\lambda-1)}2\alpha(\lambda-1)} {}_2F_1(1-\lambda, 2-2\lambda; 2-\lambda; -e^{2\alpha x}) + \frac{\sqrt{\pi}\Gamma(2-\lambda)}{2\alpha(\lambda-1)\Gamma(\frac{3}{2}-\lambda)}. \end{aligned} \quad (41)$$

This function is also odd and takes arbitrary positive values for $x > 0$ and arbitrary negative values for $x < 0$. Using the asymptotic behaviour of the hypergeometric functions (16), it is very easy to prove that the limit $\alpha \rightarrow \infty$ of (41) is

$$\lim_{\alpha \rightarrow \infty} L(\lambda, \alpha, x) = \left(\frac{e^{g|x|} - 1}{g} \right) \operatorname{sgn} x \quad (42)$$

where $\operatorname{sgn} x$ denotes the function sign of x , and we have used the fact that α, λ and g are related through equation (2). We will use $L(\lambda, \alpha, x)$ to define the function

$$\Omega_{\xi}^-(\lambda, \alpha, x) := \frac{(\cosh \alpha x)^{2(\lambda-1)}}{1 - \xi L(\lambda, \alpha, x)} = -\frac{1}{\xi} \frac{d}{dx} \ln[1 - \xi L(\lambda, \alpha, x)] \quad (43)$$

from which the following expression for β_{ξ}^- in (27) is obtained:

$$\beta_{\xi}^-(\alpha, x) = \alpha(\lambda - 1) \tanh \alpha x + \xi \Omega_{\xi}^-(\lambda, \alpha, x) = \frac{d}{dx} \ln[(\cosh \alpha x)^{1-\lambda} \Omega_{\xi}^-(\lambda, \alpha, x)]. \quad (44)$$

Using this expression and (20) we compute the new SUSY partner potential

$$\begin{aligned} \tilde{V}_{\xi}^-(\alpha, x) &= -\alpha^2 \left(1 + \frac{g}{2\alpha} - \sqrt{1 + \frac{2g}{\alpha}} \right) \frac{1}{\cosh^2 \alpha x} \\ &\quad + 4\alpha(\lambda - 1)\xi \Omega_{\xi}^-(\lambda, \alpha, x) \tanh \alpha x + 2(\xi \Omega_{\xi}^-(\lambda, \alpha, x))^2. \end{aligned} \quad (45)$$

Due to the behaviour of $L(\lambda, \alpha, x)$, it is quite clear that for any choice of $\xi \neq 0$ the function $\Omega_{\xi}^-(\lambda, \alpha, x)$ presents a singular point, and therefore the potential $\tilde{V}_{\xi}^-(\alpha, x)$ is always singular, in contradistinction to the case of $\tilde{V}_{\zeta}^+(\alpha, x)$ considered before. The presence of the singularity

suggests that the results could be interpreted according to the method developed in [23]: the SUSY partner potentials are not directly related by isospectrality to the original potential $V(\alpha, x)$, but to a different problem consisting of this modified Pöschl–Teller potential plus an infinite barrier potential placed precisely at the position where $\tilde{V}_\xi^-(\alpha, x)$ has its singular point.

The case $\xi = 0$ gives the particular solution $\tilde{V}_0^-(\alpha, x)$, which is free of singularities (see Cooper *et al* in [8]). The SUSY partner potentials (45) correspond to a factorization energy $\epsilon^- = E_0$. Therefore, there is an eigenstate $\hat{\psi}$ of H annihilated by the intertwining operator $A_0^- \hat{\psi}(x) = 0$, which is precisely (8), the ground state $\psi_0(x)$ of H . The eigenfunctions of \tilde{H}_0^- are then given by

$$\tilde{\psi}_n^-(x) = (E_n - \epsilon^-)^{-1/2} A_0^- \psi_n(x) \quad n = 1, 2, \dots \tag{46}$$

In the present case the possible missing state is not square integrable, and therefore has no physical meaning as an eigenfunction of \tilde{H}_0^- with eigenvalue ϵ^- . The spectrum of \tilde{H}_0^- is given simply by $\{E_n; n = 1, 2, \dots\}$. Remark that, as in the previous case, this new Hamiltonian is not strictly isospectral to H either, although the reason is just the opposite: now the SUSY process eliminates one eigenstate of H , while in the previous situation a new state was created, but keeping the initial spectrum. Hence, we have another example of unbroken SUSY encoded in the spectrum of the couple H and \tilde{H}_0^- .

It is interesting to evaluate the limit of the previous results for $\alpha \rightarrow \infty$, in order to do that, we first analyse the asymptotic behaviour of the factorization energy (24):

$$\epsilon^- = E_0 \sim -\left(\frac{g}{2}\right)^2 + \frac{g^3}{4\alpha} + O\left(\frac{1}{\alpha}\right)^2. \tag{47}$$

For $0 < \alpha < g/4$ the SUSY potential $\tilde{V}_0^-(\alpha, x)$ in (45) is always attractive; on the other hand, when $\alpha > g/4$ the potential becomes always repulsive; finally, when $\alpha = g/4$ the potential vanishes identically. This change of the attractive or repulsive character of $\tilde{V}_0^-(\alpha, x)$ according to the values of α has a *strong* physical meaning. It is related to the fact that in the SUSY process, the ground state level E_0 is eliminated from the spectrum of \tilde{H}_0^- , always being a member of the spectrum of H , irrespective of the value of α . In particular, when $\alpha > g/4$ the potential $V(\alpha, x)$ has only this bound state (see the comment after equation (6)), and, therefore, in the same interval the new potential $\tilde{V}_0^-(\alpha, x)$ has no bound state at all. This behaviour is illustrated in figure 3, where we plotted three members of the original family (1) of modified Pöschl–Teller potentials $V(\alpha, x)$ (the three thick curves), and also, using thin curves, their corresponding SUSY partner potentials $\tilde{V}_0^-(\alpha, x)$ given by (45). The values of the parameters are indicated in the caption.

An important detail to be stressed is that in the limit $\alpha \rightarrow \infty$ the potential $\tilde{V}_0^-(\alpha, x)$ of (45) has a well defined behaviour (unlike the situation for $\tilde{V}_0^+(\alpha, x)$): it becomes the delta barrier

$$\lim_{\alpha \rightarrow \infty} \tilde{V}_0^-(\alpha, x) = +g\delta(x) \quad g > 0. \tag{48}$$

The difference between (48) and the limit in equation (11) is quite remarkable.

Another interesting point to be considered is the analysis of the limit $\alpha \rightarrow \infty$ of (45), which can be evaluated even for the case $\xi \neq 0$. We already have all the information needed to write down this result, indeed: $\alpha(\lambda - 1) \rightarrow g/2$, $\tanh \alpha x \rightarrow \text{sgn } x$, plus equations (48), (42) and (43). We get the following:

$$\lim_{\alpha \rightarrow \infty} \beta_\xi^-(\alpha, x) = \frac{g}{2} \text{sgn } x + \xi \frac{e^{g|x|}}{1 - \xi \left(\frac{e^{g|x|}-1}{g}\right) \text{sgn } x} \tag{49}$$

$$\lim_{\alpha \rightarrow \infty} \tilde{V}_\xi^-(\alpha, x) = g\delta(x) + \frac{2g\xi e^{g|x|} \text{sgn } x}{1 - \xi \left(\frac{e^{g|x|}-1}{g}\right) \text{sgn } x} + \frac{2\xi^2 e^{2g|x|}}{\left[1 - \xi \left(\frac{e^{g|x|}-1}{g}\right) \text{sgn } x\right]^2}. \tag{50}$$

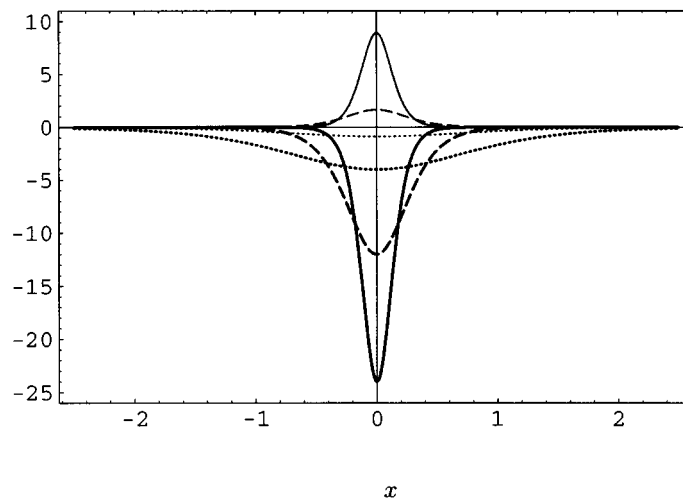


Figure 3. Different Pöschl–Teller potentials $V(\alpha, x)$ from equation (1) (the three thicker curves), and their corresponding SUSY partner potentials $\tilde{V}_0^-(\alpha, x)$ given by equation (45) (the three thinner curves). The values of the parameters are the following: dotted curves $\alpha = 1$, $\lambda = 2.562$, dashed curves $\alpha = 3$, $\lambda = 1.758$, and solid curves $\alpha = 6$, $\lambda = 1.457$. In all cases the parameter ξ is taken to be zero. The initial Pöschl–Teller potentials are always negative; their SUSY partners are less negative (dotted curve) or even become positive (dashed and solid curves).

Again, we have used the fact that the parameters α , λ and g are related through (2). Observe that after the limit process, we obtain a function with three different discontinuities:

- (1) The function blows up like $2(x - x_s)^{-2}$ at the singular point

$$x_s = \frac{\operatorname{sgn}\xi}{g} \ln \left(1 + \frac{g}{|\xi|} \right).$$

- (2) At the origin, we get a divergence of the type $+g\delta(x)$.
 (3) There is a finite jump discontinuity at the origin, due to the presence of the sign function in (50).

These remarks are clearly illustrated in figure 4, where we have plotted the limit case (without the delta distribution at the origin), plus one intermediate case. The dotted curve represents the plot of one of the SUSY partners of the modified Pöschl–Teller potential (45), for the following values of the parameters: $\alpha = 1.9$, $\lambda = 1.216$ (or equivalently $g = 1$), $\xi = -0.05$. The solid curve represents the SUSY Dirac delta potential for $g = 1$ and $\xi = -0.05$, and is a limiting case of the dotted curve when $\alpha \rightarrow \infty$ (the delta contribution comes out from the limiting behaviour of the dotted hump).

3.3. The connection with the SUSY Dirac delta potential

Let us consider now the intertwining relationship (19) for the Hamiltonian associated with the Dirac delta well potential $V_D(x) = -g\delta(x)$. Equations (20) and (21) also hold, with $V_D(x)$ as the known potential and the potential \tilde{V} to be determined. The relevant Riccati equation to be solved is now

$$\beta^2 - \beta' = -g\delta(x) - \sigma \quad (51)$$

where σ is the factorization energy in this case. Note that from the mathematical point of view it is a differential equation which includes a distribution. Therefore the solution could have

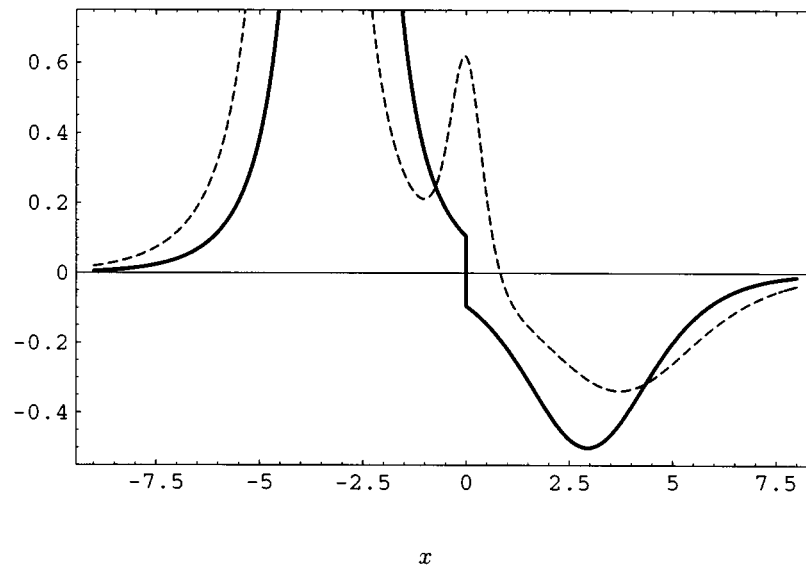


Figure 4. The dotted curve shows the plot of a member of the family of SUSY partner Pöschl–Teller potentials $\tilde{V}_\xi^-(\alpha, x)$ given in equation (45), for the following values of the parameters: $\alpha = 1.9$, $\lambda = 1.216$ (or equivalently $g = 1$), and $\xi = -0.05$. The solid curve represents the SUSY partner Dirac delta potential for $g = 1$ and $\xi = -0.05$, and is obtained from the dotted curve when the limit $\alpha \rightarrow \infty$ is considered. For this curve there are two remarkable details: first, the divergent term $g\delta(x)$ has not been represented and, second, the potential has a discontinuity at $x = 0$.

some discontinuity. It is possible to find a particular solution of (51), for a particular value of the constant σ , in terms of the sign function, indeed $\beta_0(x) = (g/2) \operatorname{sgn} x$ for $\sigma = -(g/2)^2$. This function satisfies the differential equation almost everywhere, i.e., for every real value of x , except for $x = 0$. Then, the general solution can be found by using the standard technique of transforming the nonlinear Riccati differential equation into a linear one. The final result is the following:

$$\beta_\omega(x) = \frac{g}{2} \operatorname{sgn} x - \frac{d}{dx} \ln \left(1 - \omega \int_0^x e^{g|y|} dy \right) = \frac{g}{2} \operatorname{sgn} x + \frac{\omega e^{g|x|}}{1 - \omega \left(\frac{e^{g|x|} - 1}{g} \right) \operatorname{sgn} x}. \quad (52)$$

But this is precisely the result previously obtained in (49) if we identify the two parameters $\omega = \xi$. Obviously, the potential coming out from this function will be exactly the same as (50). Therefore, we have also been able to obtain the general SUSY partners of the Dirac delta distribution as a byproduct of the general results derived for the modified Pöschl–Teller potential in the previous section. A different approach to the general solution ($\omega = \xi \neq 0$) has been given in [26]. We would like to insist on the fact that in the singular potential case ($\xi \neq 0$) the SUSY problem is related to a modification of the initial potential resulting from adding an infinite barrier placed at the singularity (see [23]).

Let us now comment a little further on the results for the Dirac delta SUSY partners. The particular case obtained from (50) by making $\xi = 0$

$$\lim_{\alpha \rightarrow \infty} \tilde{V}_0^-(\alpha, x) = +g\delta(x) = V_D^{\text{SUSY}}(x) \quad (53)$$

is in complete agreement with some already well known results [9]. Finally, we would like to stress one of the main results: the SUSY partners and all the relevant information for the Dirac delta potential can be obtained by taking the limit in the corresponding expressions for

the Pöschl–Teller case, due to the good behaviour of the limiting procedures. One remarkable difference is that for the delta potential there is just one SUSY partner (indeed, only one specific value of the factorization energy σ allows us to obtain the solution of the Riccati equation (51)), in contrast to the analysis done for the modified Pöschl–Teller potentials, where two different factorization energies ϵ^\pm were found.

4. The 2-SUSY modified Pöschl–Teller potential

The higher-order SUSY partners can also be determined for the potentials we considered before. A handy technique, which uses difference equations in order to construct multi-parametric families of isospectral potentials, has been recently developed in [20, 21]. In this section we shall comment briefly only on the results derived following this approach for the potentials we are dealing with in the 2-SUSY case. In the first SUSY step we use the factorization constant ϵ^+ and in the second step we use ϵ^- (the same final second-order SUSY results are obtained if the process is performed in reverse order). Hence, the 2-SUSY potential is given by

$$V_{\zeta, \xi}^{+, -}(\alpha, x) := V(\alpha, x) - 2 \frac{d}{dx} \left(\frac{\epsilon^+ - \epsilon^-}{\beta_\zeta^+(\alpha, x) - \beta_\xi^-(\alpha, x)} \right) \quad (54)$$

where $V(\alpha, x)$ is the initial modified Pöschl–Teller potential (1), ϵ^\pm are given by (24), and $\beta_\zeta^+(x)$, $\beta_\xi^-(x)$ by (32) and (44), respectively. The 2-SUSY partner depends on the two parameters ζ and ξ . This family embraces a wide variety of potentials; one of the most interesting cases is obtained by taking only the particular solutions $\zeta = \xi = 0$:

$$V_{0,0}(\alpha, x) = -\frac{g\alpha}{2 \cosh^2 \alpha x} + \frac{2\alpha^2}{\sinh^2 \alpha x} = -\alpha^2 \frac{\lambda(\lambda-1)}{\cosh^2 \alpha x} + \frac{2\alpha^2}{\sinh^2 \alpha x}. \quad (55)$$

This kind of solution is a particular case of a more general form of the modified Pöschl–Teller potential [2, 5]

$$V(x) = \alpha^2 \left(\frac{\kappa(\kappa+1)}{\sinh^2 \alpha x} - \frac{\lambda(\lambda-1)}{\cosh^2 \alpha x} \right) \quad (56)$$

precisely for the value $\kappa = 1$. Note that (55) does not have a well defined limit when $\alpha \rightarrow \infty$, and therefore it is not possible to find a 2-SUSY partner for the Dirac delta connected with a 2-SUSY partner of the modified Pöschl–Teller potential. Observe that this fact was implicit from the beginning, because only one factorization energy σ was found for the δ , while for the modified Pöschl–Teller we were able to find two different factorization energies ϵ^+ , ϵ^- .

5. Final remarks

Due to the relevance of solvable SUSY quantum mechanical models as toy examples for higher-dimensional quantum field theories, and also because of their use in solid state physics, we analysed in detail the supersymmetry associated with the modified Pöschl–Teller potential, which appears in many interesting physical situations, for example in the nonrelativistic limit of the sine–Gordon equation, in connection with a two-body force of Dirac delta type, when studying integrable many-body systems in one dimension, or when considering two-dimensional SUSY quantum field theories.

In this paper we have constructed two new one-parametric families of exactly solvable potentials $\tilde{V}_\zeta^+(\alpha, x)$ and $\tilde{V}_\xi^-(\alpha, x)$ related to the modified Pöschl–Teller potential $V(\alpha, x)$ by the one-parametric superpotentials $\beta_\zeta^+(\alpha, x)$ and $\beta_\xi^-(\alpha, x)$, respectively. They represent two different cases of unbroken supersymmetry, and they reduce to some previously published

results for $\zeta = 0$ or $\xi = 0$. A relevant trait of our results is that, for specific values of ζ and α , the members of the family $\tilde{V}_\zeta^+(\alpha, x)$ are free of singularities, which is, as far as we know, a fact unnoticed in the literature. On the other hand, the family $\tilde{V}_\xi^-(\alpha, x)$, with $\xi \neq 0$, embraces only singular potentials which have to be considered very carefully, because they are not SUSY partners of the initial potential (1), but of the initial potential plus an infinite barrier at the singularity [23].

The connection between the modified Pöschl–Teller and Dirac delta potentials was established, and we were able to construct the SUSY partner of the delta potential. The main remark is that only the singular family $\tilde{V}_\xi^-(\alpha, x)$ can be used to approximate an attractive delta potential in terms of the limiting procedure discussed in the paper. The explanation can be given in terms of the SUSY process: our particular solution of the Riccati equation (51) represents the superpotential $\beta_0(x) = (g/2) \operatorname{sgn} x$ usually derived for the attractive delta potential [9]. Therefore, the SUSY partner potential of the Dirac delta well is a Dirac delta barrier and, because this last potential has no bound states, the corresponding SUSY system has to present unbroken SUSY; in other words, the SUSY process eliminates the only bound state of the delta well in order to satisfy the Witten index condition for unbroken SUSY. The same holds for the case $\omega = \xi \neq 0$. The energy level at $\epsilon^- = E_0$ is destroyed and it does not play any role in the limit $\alpha \rightarrow \infty$ for the spectrum of $\tilde{V}_\xi^-(\alpha, x)$ (remember that $\lim_{\alpha \rightarrow \infty} E_0 = E_\delta$). As the other energy levels disappear, from the spectrum of $\tilde{V}_\xi^-(\alpha, x)$ when taking this limit, then the final potential has no bound states. The result is in complete agreement with the direct calculation. For $\tilde{V}_\zeta^+(\alpha, x)$, the situation is different because the ground energy level ϵ^+ diverges as $-\alpha^2$ when $\alpha \rightarrow \infty$, and the potential has no physically interesting limit.

Acknowledgments

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